

# Isotropic self-similar Markov processes

Ming Liao<sup>a,\*</sup>, Longmin Wang<sup>b</sup>

<sup>a</sup> *Department of Mathematics, Auburn University, Auburn, AL 36849, USA*

<sup>b</sup> *School of Mathematical Sciences, Nankai University, Tianjin, China*

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## Abstract

We show that an isotropic self-similar Markov process in  $\mathbb{R}^d$  has a skew product structure if and only if its radial and angular parts do not jump at the same time.

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## 1. Introduction and main result

It is well known that a Brownian motion in  $\mathbb{R}^d$  ( $d \geq 2$ ) has a skew product structure, that is, it may be expressed as a product of its radial process and a time changed spherical Brownian motion. Moreover, the radial process is a Bessel process and is independent of the spherical Brownian motion, and the time change is adapted to the radial process. This decomposition is naturally related to the invariance of the Brownian motion under the group  $O(d)$  of orthogonal transformations on  $\mathbb{R}^d$ . More generally, Galmarino [3] proved that a continuous isotropic or  $O(d)$ -invariant Markov process in  $\mathbb{R}^d$  is also a skew product of its radial motion and an independent spherical Brownian motion with a time change. Pauwels and Rogers [11] and Liao [10] extended these results to more general settings.

Because any continuous isotropic Markov process has a skew product structure, it is therefore natural to consider a similar skew product for discontinuous isotropic Markov processes.

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\* Corresponding author.

E-mail addresses: [liaomin@auburn.edu](mailto:liaomin@auburn.edu) (M. Liao), [wanglm@nankai.edu.cn](mailto:wanglm@nankai.edu.cn) (L. Wang).

Graversen and Vuolle-Apiala [4] discussed a skew product for isotropic  $\alpha$ -self-similar Markov processes, which include the purely discontinuous symmetric  $(1/\alpha)$ -stable processes. Their main result says that after a time change due to [8,7], the radial process and the angular process are respectively multiplicatively invariant and  $O(d)$ -invariant Markov processes, and are independent. This leads to a skew product structure similar to that of a Brownian motion. However, as will be shown later, the independence part of this interesting result holds only under a rather restrictive condition, which excludes for example the symmetric  $(1/\alpha)$ -stable processes for  $\alpha > 1/2$ . We note that the proof of Proposition 2.4 in [4] has an error in the conditional expectation argument.

The aim of this paper is to clarify this rather important point. We will show that an isotropic  $\alpha$ -self-similar Markov process has a skew product structure if and only if its radial and angular parts do not jump at the same time.

After the completion of this paper, we found the independence error in [4] has been noted in [1]. They used the correct part of [4] in their work, but did not pursue the independence problem.

We now describe our setup more precisely. All processes considered in this paper are assumed to have càdlàg paths (right continuous paths with left limits). Let  $x_t$  be a (time homogeneous) Markov process in  $\mathbb{R}^d$ ,  $d \geq 2$ , with transition function  $P_t$  satisfying the usual simple Markov property. We will allow process  $x_t$  to have a possibly finite life time, and as usual let  $P_x$  denote the distribution of process starting at  $x$  on the canonical space of càdlàg paths with possibly finite life times.

The restriction of process  $x_t$  on  $E = \mathbb{R}^d - \{0\}$ , defined before reaching the hitting time of origin 0, is also a Markov process. For simplicity, this process together with its transition function and distribution are still denoted by  $x_t$ ,  $P_t$  and  $P_x$ .

The process  $x_t$  in  $\mathbb{R}^d$  is said to be *isotropic* or  *$O(d)$ -invariant* if

$$P_t(\phi(x), \phi(B)) = P_t(x, B) \quad (1)$$

for any  $\phi \in O(d)$ ,  $x \in \mathbb{R}^d$  and Borel subset  $B \subset \mathbb{R}^d$ . This is equivalent to saying that process  $\phi(x_t)$  with  $x_0 = x$  has the same distribution as process  $x_t$  with  $x_0 = \phi(x)$ .

The process  $x_t$  is said to be  *$\alpha$ -self-similar*, or  *$\alpha$ -s.s.* in short, for some constant  $\alpha > 0$ , if

$$P_{\lambda t}(x, B) = P_t(\lambda^{-\alpha}x, \lambda^{-\alpha}B) \quad (2)$$

for any  $\lambda > 0$ ,  $x \in \mathbb{R}^d$  and  $B \subset \mathbb{R}^d$ . This is equivalent to saying that process  $x_{\lambda t}$  with  $x_0 = x$  has the same distribution as process  $\lambda^\alpha x_t$  with  $x_0 = \lambda^{-\alpha}x$ .

It is clear that if  $x_t$  is  $O(d)$ -invariant and/or  $\alpha$ -self-similar, so is its restriction to  $E$ .

In what follows, we will exclusively consider an isotropic Markov process  $x_t$  in  $E$ . For  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , we denote by  $|x| = \sqrt{x_1^2 + \dots + x_d^2}$  the radial part of  $x$ . Let  $r_t = |x_t|$  and  $\theta_t = x_t/r_t$  be the radial and angular parts of  $x_t$ . It is easy to show using the  $O(d)$ -invariance (see for example [10]) that  $r_t > 0$  is a 1-dim Markov process with transition function  $R_t$  given by

$$R_t f(r) = P_t(f \circ \pi_1)(x)$$

for  $r > 0$  and Borel function  $f$  on  $(0, \infty)$ , where  $\pi_1: E \rightarrow (0, \infty)$  is the natural projection given by  $y \mapsto |y|$  and  $x$  is any point of  $E$  with  $r = |x|$ . As the angular process  $\theta_t$  lives in the unit sphere  $S^{d-1}$  which is invariant under the action of  $O(d)$ , one would expect that it should inherit the  $O(d)$ -invariance of  $x_t$  in some sense. This leads to the following definition of a skew product structure.

**Definition.** Let  $x_t$  be an isotropic Markov process in  $E$ . We say that  $x_t$  has a *skew product structure* if  $x_t = r_t \xi_{A_t}$ , where  $A_t$  is a continuous and strictly increasing process with  $A_0 = 0$ , adapted to the radial process  $r_t$ , and  $\xi_t$  is an  $O(d)$ -invariant Markov process in  $S^{d-1}$  and is independent of process  $r_t$ .

Because  $O(d)$  acts on  $S^{d-1}$  transitively,  $S^{d-1}$  may be regarded as a homogeneous space of  $O(d)$ . Invariant Markov processes in homogeneous spaces are Feller processes, and their generators may be expressed explicitly in terms of an invariant differential operator and a Lévy measure (see Section 3 for more details), thus providing a useful tool for studying these processes.

The following is our main theorem.

**Theorem 1.** *Let  $x_t$  be an isotropic  $\alpha$ -self-similar Markov process in  $E = \mathbb{R}^d - \{0\}$  ( $d \geq 2$ ). Then  $x_t$  has a skew product structure if and only if its radial and angular parts do not jump at the same time, that is, for all  $x \in E$ ,  $P_x$ -almost surely,  $r_t$  and  $\theta_t$  cannot jump together at same time  $t$  for any  $t \geq 0$ .*

**Proof of the necessity part.** The sufficiency of the condition will be proved in Section 3. For the necessity, we assume that  $x_t$  has the skew product  $x_t = r_t \xi_{A_t}$ . Since  $r_t$  is càdlàg, by [6, Proposition I.1.32], the random set  $\{\Delta r_t \neq 0\}$  is thin in the sense that there is a sequence of stopping times  $\tau_n$  such that  $\{\Delta r_t \neq 0\} = \bigcup_n [[\tau_n]]$ , where  $\Delta r_t = r_t - r_{t-}$ , and  $[[\tau_n]]$  is the graph of  $\tau_n$ , i.e.,  $[[\tau_n]] = \{(\omega, t), t \in \mathbb{R}_+, t = \tau_n(\omega)\}$ . For any  $n \geq 1$ , the time  $A_{\tau_n}$  is measurable in process  $r_t$ , and the independence of  $r_t$  and  $\xi_t$  implies that  $A_{\tau_n}$  is independent of  $\xi_t$ . As a Feller process,  $\xi_t$  is quasi-left-continuous. In particular,  $\xi_t$  does not jump at a fixed time, and it is easy to see that  $\xi_t$  does not jump at  $A_{\tau_n}$ . This implies that the radial part  $r_t$  and the angular part  $\theta_t = \xi_{A_t}$  of  $x_t$  do not jump simultaneously.  $\square$

**Remark 1.** Note that an isotropic self-similar Markov process may not satisfy the condition in Theorem 1. The most famous examples are the symmetric  $(1/\alpha)$ -stable Lévy processes for  $\alpha > 1/2$ . Their Lévy measures are absolutely continuous on  $\mathbb{R}^d - \{0\}$ , so their radial and angular parts may jump together, and thus do not possess a skew product structure as defined above. On the other hand, we will see later that there are many isotropic  $\alpha$ -s.s. Markov processes that do possess a skew product structure.

**Remark 2.** It is evident that the proof above is also valid for a general isotropic Markov process. That is, the jump condition in Theorem 1 is also necessary for an isotropic Markov process to have a skew product structure.

The rest of this paper is devoted to proving the sufficiency part of Theorem 1. In Section 2, we will recall the time change used in [4], and we will show that  $x_t$  is  $\alpha$ -s.s. if and only if the time changed process is invariant under the scalar multiplication. The key fact is that if  $x_t$  is  $\alpha$ -s.s., then the time changed process is invariant under a transitive group on  $E$  and hence may be viewed as an invariant Markov process in a homogeneous space. Under this viewpoint, we complete the proof of our main theorem in Section 3.

## 2. Time changed processes

Let  $x_t$  be an isotropic Markov process (not necessarily  $\alpha$ -s.s.) in  $E = \mathbb{R}^d - \{0\}$  with a possibly finite life time  $\xi$ . Fix  $\alpha > 0$ . The following random time change was introduced by Lamperti [8] for  $\mathbb{R}_+$ -valued processes. Define

$$A_t = \int_0^t |x_s|^{-1/\alpha} ds, \quad (3)$$

which is a continuous and strictly increasing function for  $t < \xi$ . Its inverse  $T_t$  is given by

$$T_t = \inf\{s \geq 0; A_s \geq t\}, \quad t < A_{\xi-}. \quad (4)$$

We define a new process  $\{\bar{x}_t\}$  by  $\bar{x}_t = x_{T_t}$  for  $t < A_{\xi-}$  and  $x_t = \Delta$  otherwise, where  $\Delta$  is a cemetery point added to  $E$ . By Theorem 10.11 of [2],  $\bar{x}_t$  is also a time homogeneous Markov process with càdlàg paths. Let  $\bar{P}_t(x, B)$  be the transition function of  $\bar{x}_t$ . Note that  $\bar{x}_t$  and  $\bar{P}_t$  are also isotropic.

It is easy to show that

$$T_t = \int_0^t |\bar{x}_u|^{1/\alpha} du \quad (5)$$

for  $t < A_{\xi-}$ . That is,  $T_t$  is determined by the time changed process  $\bar{x}_t$  and is also continuous and strictly increasing. Note that  $A_t$  is the inverse of  $T_t$ . Thus we may start with an isotropic Markov process  $\bar{x}_t$  in  $E$  and recover the original process  $x_t$  as  $\bar{x}_{A_t}$ .

The process  $\bar{x}_t$  is said to be *multiplicatively invariant*, if

$$\bar{P}_t(x, B) = \bar{P}_t(\lambda x, \lambda B) \quad (6)$$

for any  $\lambda > 0$ ,  $x \in E$  and Borel subset  $B \subset E$ . This is equivalent to saying that process  $\lambda \bar{x}_t$  with  $\bar{x}_0 = x$  has the same distribution as process  $\bar{x}_t$  with  $\bar{x}_0 = \lambda x$ .

The following theorem relates the  $\alpha$ -self-similarity of  $x_t$  to the multiplicative invariance of  $\bar{x}_t$ . The multiplicative invariance of  $\bar{x}_t$  was proved by Kiu [7], but the present proof is simpler and more probabilistic, and also establishes its converse.

**Theorem 2.** *The process  $x_t$  is  $\alpha$ -s.s. if and only if the time changed process  $\bar{x}_t$  is multiplicatively invariant.*

**Proof.** For simplicity, we will work on the canonical probability space of càdlàg paths with possibly finite life time. We will also write  $x$  for a path  $x_t$  in  $E$  and  $x_\lambda$  for path  $t \mapsto x_{\lambda t}$  for  $\lambda > 0$ . To indicate the dependence on a path  $x$ , we will write  $A_t(x)$  and  $T_t(x)$  instead of  $A_t$  and  $T_t$ .

Assume that  $x_t$  is  $\alpha$ -s.s. Then  $(\lambda^\alpha x_{\lambda^{-1}t}, P_{\lambda^{-\alpha}x})$  is the same Markov process as  $(x_t, P_x)$  and consequently, under  $P_{\lambda^{-\alpha}x}$ , the distribution of  $(\lambda^\alpha x_{\lambda^{-1}t}, T_t(\lambda^\alpha x_{\lambda^{-1}}))$  equals that of  $(x_t, T_t(x))$  under  $P_x$ . Since

$$A_t(\lambda^\alpha x_{\lambda^{-1}}) = \lambda^{-1} \int_0^t |x_{\lambda^{-1}s}|^{-1/\alpha} ds = \int_0^{\lambda^{-1}t} |x_s|^{-1/\alpha} ds = A_{\lambda^{-1}t}(x),$$

we obtain that  $T_t(\lambda^\alpha x_{\lambda^{-1}}) = \lambda T_t(x)$ . Note that the processes  $\lambda^\alpha \bar{x}_t$  and  $\bar{x}_t$  are respectively measurable functionals of the processes  $(\lambda^\alpha x_{\lambda^{-1}t}, \lambda T_t(x))$  and  $(x_t, T_t(x))$  of the same form. It follows that process  $\lambda^\alpha \bar{x}_t$  with  $\bar{x}_0 = \lambda^{-\alpha}x$  has the same distribution as process  $\bar{x}_t$  with  $\bar{x}_0 = x$ . This proves the multiplicative invariance of  $\bar{x}_t$ .

Conversely, assume that  $\bar{x}_t$  is multiplicatively invariant. Then the process  $\lambda^\alpha \bar{x}_t$  with  $\bar{x}_0 = x$  has the same distribution as the process  $\bar{x}_t$  with  $\bar{x}_0 = \lambda^\alpha x$ . Let  $\bar{T}_t(\bar{x})$  denote the integral in (5) and let  $\bar{A}_t(\bar{x})$  be its inverse as a function of  $t$ . Then  $A_t(x) = \bar{A}_t(\bar{x})$ , and the distribution of  $(\lambda^\alpha \bar{x}_t, \bar{A}_t(\lambda^\alpha \bar{x}))$  with  $\bar{x}_0 = x$  equals that of  $(\bar{x}_t, \bar{A}_t(\bar{x}))$  with  $\bar{x}_0 = \lambda^\alpha x$ . Because  $\bar{T}_t(\lambda^\alpha \bar{x}) = \lambda \bar{T}_t(\bar{x})$ ,  $\bar{A}_t(\lambda^\alpha \bar{x}) = \bar{A}_{\lambda^{-1}t}(\bar{x}) = A_{\lambda^{-1}t}(x)$ . The  $\alpha$ -self-similarity of  $x_t$  now follows from a substitution of  $\bar{A}_t(\lambda^\alpha \bar{x})$  for  $t$  in  $\lambda^\alpha \bar{x}_t$ .  $\square$

As in [4], the semigroup property implies that there is a  $\gamma \geq 0$  such that  $\bar{P}_t(x, E) = e^{-\gamma t}$  for  $t \geq 0$  and  $x \in E$ . When  $\gamma > 0$ ,  $\bar{x}_t$  will have a finite life time, or equivalently,  $\bar{P}_t$  is not conservative. But we may define a new transition function  $\hat{P}_t$  by

$$\hat{P}_t(x, B) = e^{\gamma t} \bar{P}_t(x, B), \quad t \geq 0, x \in E, B \subset E.$$

Then  $\hat{P}_t$  is a conservative transition function, and the associated conservative Markov process  $\hat{x}_t$  is isotropic and multiplicatively invariant. The process  $\bar{x}_t$  is just process  $\hat{x}_t$  killed at an independent exponential time of rate  $\gamma$ .

### 3. Proof of the sufficiency part in Theorem 1

Let  $d \geq 2$  and let  $GL(d, \mathbb{R})$  be the group of the nonsingular linear transformations on  $\mathbb{R}^d$ . Let  $G$  be the similarity group of  $\mathbb{R}^d$ , that is,

$$G = \{g \in GL(d, \mathbb{R}); |gv| = |g||v| \text{ for any } v \in \mathbb{R}^d\},$$

where  $|v| = \sqrt{v_1^2 + \cdots + v_d^2}$  for  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$  and  $|g|$  is the operator norm of  $g \in GL(d, \mathbb{R})$ , that is,  $|g| = \sup_{|v|=1} |gv|$ . For  $c > 0$ , define the linear transformation  $m_c$  by  $m_c v = cv$  for  $v \in \mathbb{R}^d$ . Let  $R = \{m_c; c > 0\}$  and  $H = O(d)$ . Then  $R$  and  $H$  are both normal subgroups of  $G$ . Moreover,  $G$  is the direct product of  $R$  and  $H$ .

Note that  $G$  acts transitively on  $E = \mathbb{R}^d - \{0\}$ . Fix  $o = (0, \dots, 0, 1)$ . The subgroup of  $G$  fixing  $o$  is  $K = O(d-1)$ . We may identify  $G/K$  with  $E$  via the map  $gK \mapsto go$ ,  $H/K$  with the sphere  $S^{d-1}$  via  $hK \mapsto ho$ , and  $R$  with a ray in  $E \subset \mathbb{R}^d$  via  $r \mapsto ro$ . Note that  $E$  is diffeomorphic to the product space  $R \times S^{d-1}$ .

The reader is referred to Section 2.2 of [9] for some basic definitions about invariant Markov processes in homogeneous spaces. Let  $\mathfrak{g}$ ,  $\mathfrak{r}$ ,  $\mathfrak{h}$  and  $\mathfrak{k}$  be respectively the Lie algebras of  $G$ ,  $R$ ,  $H$  and  $K$ . There is an  $\text{Ad}(K)$ -invariant subspace  $\mathfrak{p}$  such that  $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ . Then the exponential map of  $G$  provides a natural local diffeomorphism from  $\mathfrak{r} \oplus \mathfrak{p}$  to  $E$ . Let  $n = \dim G$  and let  $\{X_1, \dots, X_n\}$  be a basis of  $\mathfrak{g}$  such that  $X_1 \in \mathfrak{r}$ ,  $X_2, \dots, X_d \in \mathfrak{p}$  and  $X_{d+1}, \dots, X_n \in \mathfrak{k}$ . Let  $\pi: G \rightarrow E$  be the map  $g \mapsto go$ . Restricted to a sufficient small neighborhood  $V$  of 0, the map

$$\phi: \mathbb{R}^d \ni y = (y_1, \dots, y_d) \mapsto \pi(e^{\sum_{j=1}^d y_j X_j}) \in E$$

is a diffeomorphism and  $y_1, \dots, y_d$  may be used as local coordinates on  $\phi(V)$ . As in Section 2.2 of [9], we may extend  $y_j$  to  $E$  such that  $y_j \in C_c^\infty(E)$  (the space of smooth functions on  $E$  with compact supports) and for any  $x \in E$ ,  $k \in K$ ,

$$\sum_{j=1}^d y_j(x) \text{Ad}(k) X_j = \sum_{j=1}^d y_j(kx) X_j. \quad (7)$$

As in Section 2, we let  $x_t$  be an isotropic  $\alpha$ -s.s. Markov process starting at  $x \in E$ . Recall that the time changed process  $\bar{x}_t$  defined before is a  $G$ -invariant Markov process in  $E$  with transition function  $\bar{P}_t$  (see Theorem 2). Thus for any  $f \in C_c^\infty(E)$  and  $x \in E$ ,

$$\bar{P}_t f(x) = \bar{P}_t(f \circ g)(o),$$

where  $g \in G$  is chosen to satisfy  $x = go$ . As an easy consequence,  $\bar{P}_t$  is a  $G$ -invariant Feller semigroup on  $E$ .

Let  $L$  be the generator of  $\bar{x}_t$  with domain  $\text{Dom}(L)$ . An explicit formula for the generator of an invariant Markov process in a homogeneous space was obtained by Hunt [5]. By Theorem 2.1 of [9], which is a more convenient version of Hunt's formula,  $\text{Dom}(L)$  contains  $C_c^\infty(E)$  and for  $f \in C_c^\infty(E)$ ,

$$Lf(o) = Tf(o) + \int_E \left[ f(x) - f(o) - \sum_{j=1}^d y_j(x) \frac{\partial}{\partial y_j} f(o) \right] \Pi(dx), \quad (8)$$

where  $T$  is a  $G$ -invariant diffusion generator and  $\Pi$  is a  $K$ -invariant Lévy measure on  $E$ . There exist a  $d \times d$  non-negative definite symmetric matrix  $(a_{ij})$  and constants  $c_i$  such that for  $f \in C_c^\infty(E)$ ,

$$Tf(o) = \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i^l X_j^l (f \circ \pi)(e) + \sum_{i=1}^d c_i X_i^l (f \circ \pi)(e), \quad (9)$$

where  $X_i^l$  is the left invariant vector field on  $G$  determined by  $X_i$ . Moreover, the coefficients  $a_{ij}$  and  $c_i$  satisfy

$$a_{ij} = \sum_{p,q=1}^d a_{pq} b_{ip}(k) b_{jq}(k) \quad \text{and} \quad c_i = \sum_{p=1}^d c_p b_{ip}(k), \quad \forall k \in K, \quad (10)$$

where the orthogonal matrix  $(b_{ij}(k))$  is determined by  $\text{Ad}(k)X_j = \sum_{i=1}^d b_{ij}(k)X_i$  for  $j = 1, \dots, d$ .

Since  $R$  commutes with  $H$  and  $\mathfrak{p}$  is  $\text{Ad}(K)$ -invariant,  $\text{Ad}(k)X_1 = X_1$  and  $\text{Ad}(k)X_i \in \mathfrak{p}$  for  $i \geq 2$  and  $k \in K$ . Thus  $b_{11}(k) = 1$ ,  $b_{i1}(k) = b_{1i}(k) = 0$  for  $i \geq 2$ . Then (10) implies that

$$a_{i1} = \sum_{p=2}^d a_{p1} b_{ip}(k), \quad a_{1i} = \sum_{q=2}^d a_{1q} b_{iq}(k), \quad \text{and} \quad c_i = \sum_{p=2}^d c_p b_{ip}(k), \quad 2 \leq i \leq d.$$

In other words, the vectors  $X = \sum_{i=2}^d a_{i1} X_i$  and  $Y = \sum_{i=2}^d a_{1i} X_i$  are invariant under the action of  $\text{Ad}(k)$  for all  $k \in K$ , which implies that  $X = Y = 0$ . Hence  $a_{i1} = a_{1i} = 0$  for  $2 \leq i \leq d$ . The operator  $T_2$  defined by

$$T_2 f(o) = \frac{1}{2} \sum_{i,j=2}^d a_{ij} X_i^l X_j^l (f \circ \pi)(e) + \sum_{i=2}^d c_i X_i^l (f \circ \pi)(e), \quad f \in C_c^\infty(E)$$

may be viewed as an  $H$ -invariant diffusion generator on the sphere  $S^{d-1} = H/K$ . It is well known that there is a constant  $c \geq 0$  such that  $T_2 = c\Delta$ , where  $\Delta$  is the Laplace–Beltrami operator on  $S^{d-1}$ . We define the diffusion generator  $T_1$  by

$$T_1 f(o) = \frac{1}{2} a_{11} X_1^l X_1^l (f \circ \pi)(e) + c_1 X_1^l (f \circ \pi)(e), \quad f \in C_c^\infty(E).$$

Note that operator  $T_1$  acts along  $R$ . We have proved that  $T = T_1 + T_2$  in the sense that

$$Tf(r, \theta) = (T_1 f(\cdot, \theta))(r) + (T_2 f(r, \cdot))(\theta)$$

for  $r \in R$  and  $\theta \in S^{d-1}$ .

Let  $\pi_1$  (resp.  $\pi_2$ ) be the projection from  $E$  onto  $R$  (resp.  $S^{d-1}$ ). Then for  $x \in E$ ,  $\pi_1(x)$  (resp.  $\pi_2(x)$ ) may be identified with  $|x|$  (resp.  $x/|x|$ ). Let  $\rho_t = \pi_1(\bar{x}_t)$  and  $\xi_t = \pi_2(\bar{x}_t)$ . By the  $O(d)$ -invariance of  $\bar{x}_t$ ,  $\rho_t$  is a Lévy process on  $R$  starting at  $\pi_1(x)$  and  $\xi_t$  is an  $O(d)$ -invariant Feller process on the sphere  $S^{d-1}$  starting at  $\pi_2(x)$ .

**Proposition 1.**  $\rho_t$  and  $\xi_t$  are independent if and only if the Lévy measure  $\Pi$  of  $\bar{x}_t$  is concentrated on  $R \cup S^{d-1}$ , where  $R$  and  $S^{d-1}$  are regarded as subsets of  $E \cong R \times S^{d-1}$ .

**Proof.** Assume that  $\rho_t$  and  $\xi_t$  are independent. Let  $f_1$  (resp.  $f_2$ ) be a smooth function on  $R$  (resp.  $S^{d-1}$ ) vanishing near  $o$ . Let  $f(x) = f_1(\pi_1(x))f_2(\pi_2(x))$ . Then by (8),  $\Pi(f) = \int_E f(x)\Pi(dx) = Lf(o)$ . From the independence of  $\rho_t$  and  $\xi_t$ , we have that

$$Lf(o) = \lim_{t \rightarrow 0} \frac{E[f(\bar{x}_t)]}{t} = \lim_{t \rightarrow 0} \frac{E[f_1(\rho_t)]E[f_2(\xi_t)]}{t}.$$

It follows that  $\Pi(f) = 0$  since  $E[f_1(\rho_t)] = tO(t)$  and  $E[f_2(\xi_t)] = tO(t)$  as  $t \rightarrow 0$ .

Now fix a point  $x = (r, \theta) \in E$  such that  $r$  and  $\theta$  are not the point  $o$ . We may choose positive functions  $f_1$  on  $R$  and  $f_2$  on  $S^{d-1}$  satisfying the above conditions and additionally, we assume that  $f_1 = 1$  near  $r$  and that  $f_2 = 1$  near  $\theta$ . Then there exists a neighborhood  $U$  of  $(r, \theta)$  such that  $\Pi(U) \leq \Pi(f_1 f_2) = 0$ . Hence  $(r, \theta)$  is not contained in the support of  $\Pi$ . It follows that  $\text{supp } \Pi \subset R \cup S^{d-1}$ .

Conversely, let  $\Pi = \Pi_1 + \Pi_2$  be such that  $\Pi_1$  and  $\Pi_2$  are respectively Lévy measures on  $R$  and  $S^{d-1}$ , regarded as measures on  $E$  supported by  $R$  and  $S^{d-1}$ . For  $i = 1, 2$ , let  $L_i$  be generators with diffusion parts  $T_i$  and Lévy measures  $\Pi_i$ . Our computation shows that  $L = L_1 + L_2$  at point  $o$ , and by the  $G$ -invariance of the three operators,  $L = L_1 + L_2$  on  $E$ . Note that when restricted to  $R$  (resp.  $S^{d-1}$ ),  $L_1$  (resp.  $L_2$ ) is the generator of  $\rho_t$  (resp.  $\xi_t$ ). Let  $\tilde{\rho}_t$  be a Lévy process in  $R$  with generator  $L_1$  and let  $\tilde{\xi}_t$  be an  $O(d)$ -invariant Markov process in  $S^{d-1}$  with generator  $L_2$ , and let them be independent. Then  $\tilde{x}_t = (\tilde{\rho}_t, \tilde{\xi}_t)$  is a  $G$ -invariant Markov process in  $E$  with generator  $L = L_1 + L_2$ . By the uniqueness in Theorem 2.1 of [9], the processes  $\tilde{x}_t = (\tilde{\rho}_t, \tilde{\xi}_t)$  and  $\bar{x}_t = (\rho_t, \xi_t)$  have the same distribution because they have the same generator. This shows that  $\rho_t$  and  $\xi_t$  are independent.  $\square$

**Proof of sufficiency in Theorem 1.** Now we assume that the radial and angular parts of  $x_t$  do not jump at same time. It is obvious that the time change  $\bar{x}_t = x_{T_t}$  does not change the directions of jumps. Thus the Lévy measure of  $\bar{x}_t$  is concentrated on the radial and angular axes. By Proposition 1,  $\rho_t = \pi_1(\bar{x}_t)$  and  $\xi_t = \pi_2(\bar{x}_t)$  are independent, and  $\bar{x}_t = \rho_t \xi_t$ . Recall that  $A_t = \int_0^t |x_s|^{-1/\alpha} ds$  is the inverse of  $T_t = \int_0^t (\rho_s)^{1/\alpha} ds$  and  $x_t = \bar{x}_{A_t}$ . Then  $x_t = r_t \xi_{A_t}$ , where  $r_t = |x_t| = \rho_{A_t}$  is an  $\alpha$ -self-similar process on  $(0, \infty)$  and is independent of  $\xi_t$ . Thus the skew product structure of  $x_t$  is established.  $\square$

**Remark 3.** Our proof shows that the time change by  $A_t$  provides a 1-1 correspondence between isotropic  $\alpha$ -s.s. Markov processes and  $G$ -invariant Markov processes in  $E$ . Thus, given any Lévy measure supported by  $R \cup S^{d-1}$ , there is a unique isotropic  $\alpha$ -s.s. Markov process in  $E$  that possesses a skew product structure. We also note that any isotropic  $\alpha$ -s.s. Markov process has the strong Markov property, because the strong Markov property is possessed by the time changed process and is preserved by the inverse time change.

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